CONSTRUCTION OF PERIODIC SOLUTIONS OF AUTONOMOUS SYSTEMS WITH ONE DEGREE OF FREEDOM IN THE CASE OF ARBITRARY REAL ROOTS OF THE EQUATION FOR THE BASIC AMPLITUDES

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The method for the construction of periodic solutions of autonomous systems with one degree of freedom is worked out in detail for the case of simple nonzero roots of the equation for the basic amplitudes [1,2]. The present paper considers the general case when the roots of the above-mentioned equation, being real and nonnegative numbers, can have any multiplicity.

1. Consider a nonlinear vibrating system with one degree of freedom

$$\frac{d^2x}{dt^2} + k^2x = \mu f\left(x, \frac{dx}{dt}, \mu\right) \tag{1.1}$$

The function $f(x, dx/dt, \mu)$ will be considered analytic with respect to its arguments in a certain domain. The quantity μ is a small parameter.

Introduce a new independent variable $\tau = kt$. Then equation (1.1) assumes the form

$$\frac{d^2x}{d\tau^2} + x = \frac{\mu}{k^2} f\left(x, k\frac{dx}{d\tau}, \mu\right) \tag{1.2}$$

In what follows the derivatives with respect to τ will be denoted by a letter with a prime.

Since the system is autonomous, then without loss of generality one of the initial conditions can be assumed to be

$$x'(0) = 0 \tag{1.3}$$

Corresponding to this assumption the solution of the generating equation ($\mu = 0$)

$$x_0''(\tau) + x_0(\tau) = 0$$

has the form $x_0(\tau) = A_0 \cos \tau$.

We shall seek periodic solutions of the basic equation (1.2) by the method of a small parameter. Let the second initial condition be of the form

$$x(0) = A_0 + \beta \tag{1.4}$$

where β is a function of μ vanishing for $\mu = 0$. Then the unknown function x will have the form $x = x(\tau, \beta, \mu)$.

It is well-known that the period of vibrations of the autonomous system (1.2) depends on the parameter μ and may be put in the form of a sum $T = 2\pi + a$, where 2π is the period of the generating solution and a is a certain function of μ vanishing for $\mu = 0$.

Let us determine the structure of the function $x(\tau, \beta, \mu)$. Assuming that this function can be developed in a series of integral powers of the parameters β and μ, we shall have

$$x(\tau, \beta, \mu) = x_0(\tau) + B_1(\tau)\beta + C_1(\tau)\mu + B_2(\tau)\beta^2 + D_1(\tau)\beta\mu + C_2(\tau)\mu^2 + \ldots$$

The functions $B_n(\tau)$ satisfy the equation

$$B_n''(\tau) + B_n(\tau) = 0 \qquad (n = 1, 2, \ldots)$$

the initial conditions being

$$B_1(0) = 1, \quad B_1'(0) = 0, \quad B_n(0) = 0, \quad B_n'(0) = 0 \qquad (n = 2, 3, \ldots)$$

Consequently,

$$B_1(\tau) = \cos\tau, \quad B_n(\tau) = 0 \qquad (n = 2, 3, \ldots)$$

Taking this into account, the function $x(\tau, \beta, \mu)$ can be represented in the form

$$x(\tau, \beta, \mu) = A_0 \cos\tau + \beta\cos\tau + \sum_{n=1}^{\infty} \left(C_n + \frac{\partial C_n}{\partial\beta}\beta + \frac{1}{2}\frac{\partial^2 C_n}{\partial\beta^2}\beta^2 + \ldots\right)\mu^n \tag{1.5}$$

It is necessary to remember that all $C_n(\tau)$ and their derivatives with respect to β are evaluated for $\beta = \mu = 0$. From formula (1.5) it follows that the identity

$$\left(\frac{\partial^m x}{\partial\beta^m}\right)_{\mu=0} = \left(\frac{\partial^m x}{\partial A_0{}^m}\right)_{\mu=0} \tag{1.6}$$

holds.

It is obvious that this property is possessed not only by x but also by any of its derivatives with respect to τ. In particular,

$$\left(\frac{\partial^m x'}{\partial \beta^m}\right)_{\mu=0} = \left(\frac{\partial^m x'}{\partial A_0{}^m}\right)_{\mu=0} \tag{1.7}$$

Let us prove that differentiation with respect to β can be replaced by differentiation with respect to A_0 also in mixed derivatives of x with respect to β and μ, these derivatives being evaluated for the zero values of the parameters. First notice that under the assumptions made the function $f(x, kx', \mu)$ can also be developed in a double series of integral powers of β and μ.

Consider the coefficient of $\beta^m \mu^{n+1}$ in the expansion of $x(\tau, \beta, \mu)$. This coefficient can be obtained by solving the corresponding nonhomogeneous differential equation of the second order for the initial conditions equal to zero. Multiplying both sides of this solution by $m!(n+1)!$ we obtain

$$\left(\frac{\partial^{m+n+1} x}{\partial \beta^m \partial \mu^{n+1}}\right)_0 = \frac{n+1}{k^2} \int_0^\tau \left(\frac{\partial^{m+n} f}{\partial \beta^m \partial \mu^n}\right)_0 \sin(\tau - \tau_1)\, d\tau_1 \tag{1.8}$$

Assume that

$$\left(\frac{\partial^{m+n} x}{\partial \beta^m \partial \mu^n}\right)_{\beta=0,\,\mu=0} = \left(\frac{\partial^{m+n} x}{\partial A_0{}^m \partial \mu^n}\right)_{\beta=0,\,\mu=0} \tag{1.9}$$

and that a similar identity exists for x'. Let us prove that analogous formulas hold for the mixed derivative of the order $m + (n + 1)$.

Since

$$\frac{\partial^{m+n} f}{\partial \beta^m \partial \mu^n} = F_{m,n}\left(\frac{\partial x}{\partial \beta}, \frac{\partial x}{\partial \mu}, \ldots, \frac{\partial^{m+n} x}{\partial \beta^m \partial \mu^n}, \frac{\partial x'}{\partial \beta}, \frac{\partial x'}{\partial \mu}, \ldots, \frac{\partial^{m+n} x'}{\partial \beta^m \partial \mu^n}\right)$$

then

$$\left(\frac{\partial^{m+n} f}{\partial \beta^m \partial \mu^n}\right)_{\beta=0,\,\mu=0} = \left(\frac{\partial^{m+n} f}{\partial A_0{}^m \partial \mu^n}\right)_{\beta=0,\,\mu=0}$$

From the formula (1.8) and the formula obtained by differentiating both of its sides with respect to τ, the sought relations immediately follow. The identity (1.9) holds for $n = 0$, and, consequently, it will hold also for any n. Analogous identities hold for any derivatives of x with respect to τ.

Making use of these properties, the function $x(\tau, \beta, \mu)$ can be represented in the form

$$x(\tau, \beta, \mu) = A_0 \cos\tau + \beta \cos\tau + \sum_{n=1}^{\infty} \left(C_n + \frac{\partial C_n}{\partial A_0}\beta + \frac{1}{2}\frac{\partial^2 C_n}{\partial A_0{}^2}\beta^2 + \ldots\right)\mu^n \tag{1.10}$$

Consequently, in order to construct the function $x(\tau, \beta, \mu)$, it is necessary first to know the coefficients $C_n(\tau)$ of μ^n. The remaining coefficients of the double series can then be obtained by successive differentiation of $C_n(\tau)$ with respect to A_0. Denote

$$H_n(\tau) = \frac{1}{(n-1)!}\left(\frac{d^{n-1}f}{d\mu^{n-1}}\right)_{\beta=0,\ \mu=0}$$

where $df/d\mu$ is the complete partial derivative of the function $f(x, kx', \mu)$ with respect to μ. The coefficients $C_n(\tau)$ satisfy the equation

$$C_n''(\tau) + C_n(\tau) = \frac{1}{k^2} H_n(\tau)$$

the initial conditions being $C_n(0) = 0$ and $C_n'(0) = 0$. From this it follows that

$$C_n(\tau) = \frac{1}{k^2}\int_0^\tau H_n(\tau_1)\sin(\tau-\tau_1)\,d\tau_1, \quad C_n'(\tau) = \frac{1}{k^2}\int_0^\tau H_n(\tau_1)\cos(\tau-\tau_1)\,d\tau_1 \qquad (1.11)$$

The first four functions $H_n(\tau)$ are

$$H_1(\tau) = f(x_0, kx_0', \mu) = f(A_0\cos\tau, -kA_0\sin\tau, 0) \qquad (1.12)$$

$$H_2(\tau) = \left(\frac{\partial f}{\partial x}\right)_0 C_1 + \left(\frac{\partial f}{\partial x'}\right)_0 C_1' + \left(\frac{\partial f}{\partial \mu}\right)_0 \qquad (1.13)$$

$$H_3(\tau) = \frac{1}{2}\left(\frac{\partial^2 f}{\partial x^2}\right)_0 C_1^2 + \frac{1}{2}\left(\frac{\partial^2 f}{\partial x'^2}\right)_0 C_1'^2 + \frac{1}{2}\left(\frac{\partial^2 f}{\partial \mu^2}\right)_0 + \left(\frac{\partial^2 f}{\partial x\partial x'}\right)_0 C_1C_1' +$$
$$+ \left(\frac{\partial^2 f}{\partial x\partial \mu}\right)_0 C_1 + \left(\frac{\partial^2 f}{\partial x'\partial \mu}\right)_0 C_1' + \left(\frac{\partial f}{\partial x}\right)_0 C_2 + \left(\frac{\partial f}{\partial x'}\right)_0 C_2' \qquad (1.14)$$

$$H_4(\tau) = \frac{1}{6}\left(\frac{\partial^3 f}{\partial x^3}\right)_0 C_1^3 + \frac{1}{6}\left(\frac{\partial^3 f}{\partial x'^3}\right)_0 C_1'^3 + \frac{1}{6}\left(\frac{\partial^3 f}{\partial \mu^3}\right)_0 + \frac{1}{2}\left(\frac{\partial^3 f}{\partial x^2\partial x'}\right)_0 C_1^2C_1' +$$
$$+ \frac{1}{2}\left(\frac{\partial^3 f}{\partial x\partial x'^2}\right)_0 C_1C_1'^2 + \frac{1}{2}\left(\frac{\partial^3 f}{\partial x^2\partial \mu}\right)_0 C_1^2 + \frac{1}{2}\left(\frac{\partial^3 f}{\partial x'^2\partial \mu}\right)_0 C_1'^2 + \frac{1}{2}\left(\frac{\partial^3 f}{\partial x\partial \mu^2}\right)_0 C_1 +$$
$$+ \frac{1}{2}\left(\frac{\partial^3 f}{\partial x'\partial \mu^2}\right)_0 C_1' + \left(\frac{\partial^3 f}{\partial x\partial x'\partial \mu}\right)_0 C_1C_1' + \left(\frac{\partial^2 f}{\partial x^2}\right)_0 C_1C_2 + \left(\frac{\partial^2 f}{\partial x'^2}\right)_0 C_1'C_2' +$$
$$+ \left(\frac{\partial^2 f}{\partial x\,\partial x'}\right)_0 (C_1C_2' + C_1'C_2) + \left(\frac{\partial^2 f}{\partial x\partial \mu}\right)_0 C_2 + \left(\frac{\partial^2 f}{\partial x'\partial \mu}\right)_0 C_2' + \left(\frac{\partial f}{\partial x}\right)_0 C_3 + \left(\frac{\partial f}{\partial x'}\right)_0 C_3'$$
$$\cdots\cdots\cdots\cdots\cdots\cdots \qquad (1.15)$$

The subscript 0 indicates that in the partial derivatives of the function f the variables x, x' and μ must be replaced by $A_0\cos\tau$, $-A_0\sin\tau$ and 0 respectively.

2. The periodicity conditions for the function $x(\tau, \beta, \mu)$ and its first derivative with respect to τ, in conformity with the initial conditions (1.4) and (1.3), can be given in the form

$$x(2\pi + \alpha, \beta, \mu) = A_0 + \beta \qquad (2.1)$$

$$x'(2\pi + \alpha, \beta, \mu) = 0 \qquad (2.2)$$

From condition (2.2) which defines the quantity α implicitly, let us determine $\alpha = \alpha(\beta, \mu)$ in the form of a double series of integral powers of β and μ. For the determination of the coefficients of this series it is necessary to calculate the partial derivatives of α with respect to

β and μ for $\tau = 2\pi$ and $\beta = \mu = 0$.

The existence of partial derivatives of the implicit function $\alpha(\beta, \mu)$ is assured in the present case by the condition

$$x''(2\pi, 0, 0) = -A_0 \neq 0 \tag{2.3}$$

Thus the construction of a series solution for the function $\alpha(\beta, \mu)$ is possible only in the case when the amplitude of the generating solution is different from zero.

Since all derivatives of $x'(\tau, \beta, \mu)$ with respect to β are equal to zero for $\tau = 2\pi$ and $\mu = 0$, then

$$\left(\frac{\partial^m \alpha}{\partial \beta^m}\right)_{\mu=0} = 0 \qquad (m = 1, 2, \ldots)$$

On the basis of the identity (1.9) in mixed derivatives the differentiation of the function α with respect to β can be replaced by differentiation with respect to A_0, i.e.

$$\left(\frac{\partial^{m+n}\alpha}{\partial \beta^m \partial \mu^n}\right)_{\beta=0,\,\mu=0} = \left(\frac{\partial^{m+n}\alpha}{\partial A_0{}^m \partial \mu^n}\right)_{\beta=0,\,\mu=0}$$

Consequently, it is only necessary to calculate the partial derivatives of the function α with respect to μ. Calculating successively the first four derivatives, we obtain

$$\left(\frac{\partial \alpha}{\partial \mu}\right)_0 = \frac{1}{A_0} C_1'(2\pi) = N_1(2\pi) \tag{2.4}$$

$$\left(\frac{\partial^2 \alpha}{\partial \mu^2}\right)_0 = \frac{2}{A_0}\left[C_2'(2\pi) + \frac{1}{k^2} H_1(2\pi) N_1(2\pi)\right] = 2N_2(2\pi) \tag{2.5}$$

$$\left(\frac{\partial^3 \alpha}{\partial \mu^3}\right)_0 = \frac{6}{A_0}\Big\{C_3'(2\pi) + \frac{1}{k^2} H_1(2\pi) N_2(2\pi) - \Big[C_2(2\pi) + \frac{1}{3A_0} C_1'^2(2\pi) - \\ - \frac{1}{2k^2 A_0} H_1'(2\pi) C_1'(2\pi) - \frac{1}{k^2} H_2(2\pi)\Big] N_1(2\pi)\Big\} = 6N_3(2\pi) \tag{2.6}$$

$$\left(\frac{\partial^4 \alpha}{\partial \mu^4}\right)_0 = \frac{24}{A_0}\Big\{C_4'(2\pi) + \frac{1}{k^2} H_1(2\pi) N_3(2\pi) - \Big[C_2(2\pi) - \frac{1}{k^2} H_2(2\pi)\Big] N_2(2\pi) - \\ - \Big[C_3(2\pi) + \frac{1}{A_0} C_2'(2\pi) C_1'(2\pi) - \frac{1}{k^2 A_0} H_1'(2\pi) C_2'(2\pi) + \frac{2}{3k^2 A_0{}^2} H_1(2\pi) C_1'^2(2\pi) - \\ - \frac{1}{6k^2 A_0{}^2} H_1''(2\pi) C_1'^2(2\pi) - \frac{1}{k^4 A_0{}^2} H_1(2\pi) H_1'(2\pi) C_1'(2\pi) - \\ - \frac{1}{2k^2 A_0} H_2'(2\pi) C_1'(2\pi) - \frac{1}{k^2} H_3(2\pi)\Big] N_1(2\pi)\Big\} = 24N_4(2\pi) \tag{2.7}$$

. . . .

Hence for the function α we obtain the expression

$$\alpha(\beta, \mu) = \sum_{n=1}^{\infty}\left(N_n + \frac{\partial N_n}{\partial A_0}\beta + \frac{1}{2}\frac{\partial^2 N_n}{\partial A_0{}^2}\beta^2 + \ldots\right)\mu^n \tag{2.8}$$

Let us note that all quantities N_n and their derivatives with respect to A_0 are evaluated for $\tau = 2\pi$ and $\beta = \mu = 0$.

Now consider the condition (2.1). For a preliminary simplification of this condition expand the left-hand sides of (2.1) and (2.2) in series of powers of a. Multiply the equality (2.2) by a and subtract it term-by-term from the equality (2.1). We then obtain

$$x(2\pi,\ \beta,\mu)-\frac{1}{2}\alpha^2x''(2\pi,\ \beta,\mu)-\frac{1}{3}\alpha^3x'''(2\pi,\ \beta,\mu)-\frac{1}{8}\alpha^4\alpha''''(2\pi,\ \beta,\mu)-\ldots= \\ =A_0+\beta \qquad (2.9)$$

Substitute in this equality the expressions for $x(2\pi,\beta,\mu)$, $x''(2\pi,\beta,\mu)$, ... as given by formula (1.10) and for a its expression by formula (2.8). The terms of the left-hand side of this equality are products of two functions, each of which possesses the property (1.9). Terms on the same side which are independent of μ are compensated by the right-hand side. Thus the condition (2.9) can be put in the form

$$\sum_{n=1}^{\infty}\Big(M_n+\frac{\partial M_n}{\partial A_0}\beta+\frac{1}{2}\frac{\partial^2 M_n}{\partial A_0^2}\beta^2+\ldots\Big)\mu^n=0 \qquad (2.10)$$

Let us note that the quantities M_n are also evaluated for $\tau = 2\pi$ and $\beta=\mu=0$. Calculating them we obtain

$$M_1(2\pi)=C_1(2\pi) \qquad (2.11)$$

$$M_2(2\pi)=C_2(2\pi)+\frac{1}{2A_0}C_1'^2(2\pi) \qquad (2.12)$$

$$M_3(2\pi)=C_3(2\pi)+\frac{1}{A_0}\Big[C_2'(2\pi)+\frac{1}{2k^2A_0}H_1(2\pi)C_1'(2\pi)\Big]C_1'(2\pi) \qquad (2.13)$$

$$M_4(2\pi)=C_4(2\pi)+\frac{1}{2A_0}C_2'^2(2\pi)+\frac{1}{A_0}\Big[C_3'(2\pi)+\frac{1}{k^2A_0}H_1(2\pi)C_2'(2\pi)- \\ -\frac{1}{2A_0}C_2(2\pi)C_1'(2\pi)-\frac{1}{8A_0^2}C_1'^3(2\pi)+\frac{1}{6k^2A^2}H_1'(2\pi)C_1'^2(2\pi)+ \\ +\frac{1}{2k^2A_0}H_2(2\pi)C_1'(2\pi)+\frac{1}{2k^4A_0^2}H_1^2(2\pi)C_1'(2\pi)\Big]C_1'(2\pi) \qquad (2.14)$$

. . . .

3. The obtained formulas permit us to pass to the determination of the quantities a and β and to construct periodic solutions of the equation (1.2).

Assume that the quantity β can be expanded in a power series of μ, i.e.

$$\beta=\sum_{n=1}^{\infty}A_n\mu^n \qquad (3.1)$$

Substituting this series into the formula (2.10) and equating to zero the coefficients of all powers of μ, we obtain*

$$M_1(2\pi)=C_1(2\pi)=0 \qquad (3.2)$$

$$A_1\frac{\partial C_1}{\partial A_0}+M_2(2\pi)=0 \qquad (3.3)$$

* The formulas (3.2) to (3.4) coincide with the periodicity conditions (21), (26) and (31) for the functions $a_n(\tau)$ in the paper [2].

$$A_2 \frac{\partial C_1}{\partial A_0} + \frac{1}{2} A_1^2 \frac{\partial^2 C_1}{\partial A_0^2} + A_1 \frac{\partial M_2}{\partial A_0} + M_3(2\pi) = 0 \tag{3.4}$$

$$A_3 \frac{\partial C_1}{\partial A_0} + A_2\left(\frac{\partial M_2}{\partial A_0} + A_1 \frac{\partial^2 C_1}{\partial A_0^2}\right) + \frac{1}{6} A_1^3 \frac{\partial^3 C_1}{\partial A_0^3} +$$

$$+ \frac{1}{2} A_1^2 \frac{\partial^2 M_2}{\partial A_0^2} + A_1 \frac{\partial M_3}{\partial A_0} + M_4(2\pi) = 0 \tag{3.5}$$

. . . .

From the obtained condition the amplitude A_0 of the generating solution and the coefficients $A_n (n = 1, 2, \ldots)$ determining the initial displacement β of the system can be calculated in succession.

The condition (3.2) is an amplitudinal equation from which A_0 can be calculated. If A_0 is not a multiple root of the equation (3.2), then from the remaining conditions, which are linear with respect to A_1, A_2, ..., these coefficients can be calculated in succession.

If equation (3.2) has a double root, then for such a root $\partial C_1 / \partial A_0 = 0$.

In order that a periodic solution exist in the last case, it is necessary that the supplementary condition $M_2(2\pi) = 0$ be satisfied. This last fact is easily seen from the condition (3.3).

If this condition, together with that of (3.2), is identically satisfied, then $\partial M_2 / \partial A_0 = 0$. The coefficient A_1 is given by the quadratic equation

$$\frac{1}{2} A_1^2 \frac{\partial^2 C_1}{\partial A_0^2} + M_3(2\pi) = 0$$

Since the quantity A_1 must be real, then there can exist for A_1 either two values or none at all. The equation for the determination of the remaining coefficients A_2, A_3, ... will again be linear.

The case of a triple root of the amplitude equation (3.2) can be discussed in a similar way. Here, for the existence of a periodic solution, it is necessary to satisfy one additional condition $M_3(2\pi) = 0$. The coefficient A_1 in such a case is determined by an equation of degree three.

It is obvious that for a root of multiplicity n of the equation (3.2), $n - 1$ supplementary conditions will have to be satisfied, and the coefficient A_1 will be given by an equation of degree n.

Thus, in the case of multiple roots of the equation for the basic amplitudes, bifurcation of the generating solution is possible.

In order to determine the period of the solution of the equation (1.2) assume that this period can be represented in the form of a series in powers of μ,

$$T = 2\pi(1 + h_1\mu + h_2\mu^2 + \cdots) \tag{3.6}$$

Then

$$\alpha = 2\pi \sum_{n=1}^{\infty} h_n\mu^n \tag{3.7}$$

Substitute in the right-hand side of the formula (2.8) the expression (3.1) for β and replace the left-hand side by the series (3.7). Equating the coefficients of equal powers of μ on both sides of the equality we obtain*

$$h_1 = \frac{1}{2\pi} N_1 \tag{3.8}$$

$$h_2 = \frac{1}{2\pi}\left(A_1 \frac{\partial N_1}{\partial A_0} + N_2\right) \tag{3.9}$$

$$h_3 = \frac{1}{2\pi}\left(A_2 \frac{\partial N_1}{\partial A_0} + \frac{1}{2} A_1^2 \frac{\partial^2 N_1}{\partial A_0^2} + A_1 \frac{\partial N_2}{\partial A_0} + N_3\right) \tag{3.10}$$

$$h_4 = \frac{1}{2\pi}\left[A_3 \frac{\partial N_1}{\partial A_0} + A_2\left(\frac{\partial N_2}{\partial A_0} + A_1 \frac{\partial^2 N_1}{\partial A_0^2}\right) + \frac{1}{6} A_1^3 \frac{\partial^3 N_1}{\partial A_0^3} + \right.$$
$$\left. + \frac{1}{2} A_1^2 \frac{\partial^2 N_2}{\partial A_0^2} + A_1 \frac{\partial N_3}{\partial A_0} + N_4\right] \tag{3.11}$$

. . . .

Let us make an additional substitution for the independent variable by putting

$$\tau = \tau_0(1 + h_1\mu + h_2\mu^2 + \cdots) \tag{3.12}$$

and let us look for a solution of the equation (1.2) in terms of τ_0. This solution has period 2π, independent of μ. Substitute (3.12) for τ into the functions $C_n(\tau)$ and $\cos\tau$ and expand them in series of powers of μ. We then obtain

$$C_n(\tau) = C_n(\tau_0) + h_1\tau_0 C_n'(\tau_0)\mu + \left[h_2\tau_0 C_n'(\tau_0) + \frac{1}{2} h_1^2\tau_0^2 C_n''(\tau_0)\right]\mu^2 + \cdots \tag{3.13}$$

and

$$\cos\tau = \cos\tau_0 - h_1\tau_0 \sin\tau_0\mu - \left(h_2\tau_0 \sin\tau_0 + \frac{1}{2} h_1^2\tau_0^2 \cos\tau_0\right)\mu^2 -$$
$$- (h_3\tau_0 \sin\tau_0 + h_1h_2\tau_0^2 \cos\tau_0 - \frac{1}{6} h_1^3\tau_0^3 \sin\tau_0)\mu^3 - \cdots \tag{3.14}$$

Substitute into the formula (1.10) the expressions (3.13) and (3.14) for the functions $C_n(\tau)$ and $\cos\tau$ and drop the subscript 0 of τ_0. Expand the left-hand side of this formula in the form of a series in powers of the parameter μ, i.e. let

* Formulas (3.8) to (3.10) coincide with the formulas (22), (27) and (32) of the paper [2], obtained from the conditions of periodicity for the functions $b_n(\tau)$.

$$x(\tau) = x_0(\tau) + \mu x_1(\tau) + \mu^2 x_2(\tau) + \cdots \tag{3.15}$$

Equating the coefficients of equal powers of μ on both sides of the equality, we obtain

$$x_0(\tau) = A_0 \cos\tau \tag{3.16}$$

$$x_1(\tau) = A_1 \cos\tau + C_1(\tau) - h_1 A_0 \tau \sin\tau \tag{3.17}$$

$$x_2(\tau) = A_2 \cos\tau + C_2(\tau) + h_1 \tau C_1'(\tau) + A_1 \frac{\partial C_1}{\partial A_0} - \\ - h_1 A_1 \tau \sin\tau - \tfrac{1}{2} h_1^2 A_0 \tau^2 \cos\tau - h_2 A_0 \tau \sin\tau \tag{3.18}$$

$$x_3(\tau) = A_3 \cos\tau + C_3(\tau) + h_1 \tau C_2'(\tau) + A_1 \frac{\partial C_2}{\partial A_0} + h_1 A_1 \tau \frac{\partial C_2'}{\partial A_0} + \tfrac{1}{2} h_1^2 \tau^2 C_1''(\tau) + \\ + h_2 \tau C_1'(\tau) + A_2 \frac{\partial C_1}{\partial A_0} + \tfrac{1}{2} A_1^2 \frac{\partial^2 C_1}{\partial A_0^2} - h_1 A_2 \tau \sin\tau - \tfrac{1}{2} - h_1^2 A_1 \tau^2 \cos\tau - \\ - h_2 A_1 \tau \sin\tau + \tfrac{1}{6} h_1^3 A_0 \tau^3 \sin\tau - h_1 h_2 A_0 \tau^2 \cos\tau - h_3 A_0 \tau \sin\tau \tag{3.19}$$

. . . .

It is obvious that the functions $C_n(\tau)$ contain nonperiodic terms which are compensated by the terms depending on $h_1, h_2, \ldots, h_n$.

4. Consider the case of a zero root $A_0 = 0$ of the equation (3.2) for the basic amplitudes.

Let us note that one root of the amplitude equation is always zero, which may also be a multiple root of odd order. The preceding analysis, however, is not applicable in the case of a zero root.

By means of formula (3.12) introduce into equation (1.2) a new independent variable. Then we obtain

$$x'' + h^2 x = \mu \frac{h^2}{k^2} f\left(x, \frac{k}{h} x', \mu\right) \tag{4.1}$$

where

$$h = 1 + h_1 \mu + h_2 \mu^2 + \cdots \tag{4.2}$$

Let us look for a solution of (4.1) in the form of a series (3.15). The function x_1 is determined by the equation

$$x_1'' + x_1 = k^{-2} f(0, 0, 0)$$

The solution of this equation under the conditions (1.3) and (1.4) have the form

$$x_1 = P_1 \cos\tau + k^{-2} f(0, 0, 0), \qquad P_1 = A_1 - k^{-2} f(0, 0, 0)$$

For the function x_2 we have the equation

$$x_2'' + x_2 = -2h_1 x_1 + \frac{2h_1}{k^2} f(0, 0, 0) + \frac{1}{k^2}\left[\left(\frac{\partial f}{\partial x}\right)_0 x_1 + \left(\frac{\partial f}{\partial x'}\right)_0 x_1' + \left(\frac{\partial f}{\partial \mu}\right)_0\right]$$

If $A_0 = 0$ is a simple root, then from the periodicity condition for x_2 it follows that $P_1 = 0$. In the case of a multiple zero root we have the condition

$$\left[2h_1 - \frac{1}{k^2}\left(\frac{\partial f}{\partial x}\right)_0\right] P_1 = 0$$

Under the assumption that the expression in the squared brackets is equal to zero, it follows from the periodicity condition for the function x_3 that $P_1 = 0$. Hence we have that $x_1 = A_1$. From the conditions of periodicity for the immediately following functions x_n we obtain that $x_2 = A_2$, $x_3 = A_3$, and so on. Consequently,

$$x = \sum_{n=1}^{\infty} A_n \mu^n = \beta(\mu) \tag{4.3}$$

Thus the zero solution of the generating equation corresponds to the time independent solution of the complete equation.

Substitute the obtained value of x in equation (1.2). Then

$$\beta = \mu k^{-2} f(\beta,\ 0,\ \mu) \tag{4.4}$$

Expanding both sides of this equality in series of powers of μ and equating coefficients of equal power of μ, we will obtain formulas for the successive determination of the coefficients A_n.

The case of the zero root corresponds to the equilibrium of the generating system. Consequently, the equilibrium of the generating solution goes over into the equilibrium of the complete system. Only the coordinate of the equilibrium position can change.

5. Let the function $f(x, kx', \mu)$ be independent of μ and have the form $f(x, kx') = kf_1(x)x'$. In this case the expressions for $H_x(\tau)$ simplify to

$$H_2(\tau) = k\frac{\partial}{\partial\tau}[f_1(x_0)C_1] \tag{5.1}$$

$$H_3(\tau) = k\frac{\partial}{\partial\tau}\left[\frac{1}{2}f_1'(x_0)C_1^2 + f_1(x_0)C_2\right] \tag{5.2}$$

$$H_4(\tau) = k\frac{\partial}{\partial\tau}\left[\frac{1}{6}f_1''(x_0)C_1^3 + f_1'(x_0)C_1C_2 + f_1(x_0)C_3\right] \tag{5.3}$$

It can be shown that in this case $C_1'(2\pi) = 0$ and $C_2(2\pi) = 0$.

On the basis of (2.12) and (3.3) it follows that $A_1 = 0$ in the case of simple roots of the equation for the basic amplitudes. Further it can be proved that in the case of a root of multiplicity n the equality $\partial^{n-1}C_2/\partial A_0^{n-1} = 0$ holds.

6. Finally, consider some examples. 1. Let the function $f(x, kx')$ be of the form

$$f(x,\ kx') = k(\alpha + \varepsilon x + \beta x^2 + \varkappa x^3 + \gamma x^4)\,x' \tag{6.1}$$

The equation for the basic amplitudes, after rejection of the zero root, is

$$\gamma A_0^4 + 2\beta A_0^2 + 8\alpha = 0.$$

The double root of this equation is

$$A_0^2 = -\beta/\gamma, \qquad \beta^2 = 8\alpha\gamma$$

The equation for the determination of the coefficient A_1 is

$$A_1^2 + \tfrac{1}{576} A_0^4 k^{-2}\left(\tfrac{1}{128}\gamma^2 A_0^6 + \tfrac{257}{100}\varkappa^2 A_0^4 + \tfrac{41}{5}\varepsilon\varkappa A_0^2 + \tfrac{13}{2}\varepsilon^2\right) = 0$$

This equation does not possess real roots either in the case $\epsilon = \kappa = 0$ or in the general case. Thus, in the case of multiple roots of the amplitude equation, periodic solutions of the equation (1.1) do not exist if the function $f(x,kx')$ is given by the formula (6.1).

2. Let $f(x,kx')$ be of the form

$$f(x,\ kx') = k(\alpha + \beta x^2 + \gamma x^4 + \delta x^6)\,x' \tag{6.2}$$

The equation for the basic amplitudes (besides the zero root) is

$$5\delta A_0^6 + 8\gamma A_0^4 + 16\beta A_0^2 + 64\alpha = 0$$

If one considers a double root of this equation, then an additional equation

$$15\delta A_0^4 + 16\gamma A_0^2 + 16\beta = 0$$

must be considered.

Evaluating $C_1(\tau)$ we obtain that

$$C_1(\tau) = \tfrac{1}{64} A_0^5 k^{-1}\left[\tfrac{1}{48}\delta A_0^2 \sin 7\tau + \left(\tfrac{5}{24}\delta A_0^2 + \tfrac{1}{6}\gamma\right)\sin 5\tau - \right.$$
$$\left. - \left(\tfrac{3}{4}\delta A_0^2 + \tfrac{1}{2}\gamma\right)\sin 3\tau + \left(\tfrac{17}{16}\delta A_0^2 + \tfrac{2}{3}\gamma\right)\sin\tau\right]$$

For the coefficient A_1 we have the equation

$$(15\delta A_0^2 + 8\gamma)A_1^2 + \tfrac{1}{128^2} A_0^{10} k^{-2}\left(\tfrac{2117}{384}\delta^3 A_0^6 + \tfrac{1621}{144}\gamma\delta^2 A_0^4 + \right.$$
$$\left. + \tfrac{1139}{18}\gamma^2\delta A_0^2 + \tfrac{16}{9}\gamma^3\right) = 0$$

For $\delta = 0$ a periodic solution will not exist (as a consequence of Example 1.). For $\delta \neq 0$ a sufficiently narrow interval will exist in which periodic vibrations are possible. These intervals are bounded by the zeros of the polynomials, being the coefficient of $A_1{}^2$ and the free term of the equation. We have

$$-1.875A_0^2 < \gamma/\delta < -1.527A_0^2$$

The upper bound is determined only approximately. The period of vibrations up to the terms containing μ^3 is

$$T = \frac{2\pi}{k} \cdot \left[1 + \frac{1}{64^2} A_0^8 k^{-2} \left(\frac{89}{32} \delta^2 A_0^4 + \frac{23}{6} \delta\gamma A_0^2 + \frac{4}{3} \gamma^2\right) \mu^2 + \cdots\right]$$

The vibrations themselves up to terms containing μ^2 are given by the formula

$$x(\tau) = A_0 \cos\tau + [A_1 \cos\tau + C_1(\tau)]\mu + \cdots$$

where A_0, A_1 and $C_1(\tau)$ are determined by the above mentioned formulas.

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